PHYSICAL REVIEW E VOLUME 58, NUMBER 4 OCTOBER 1998

Korteweg–de Vries solitons under additive stochastic perturbations

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(Received 10 October 1997; revised manuscript received 24 April 1998)

The evolution of solitonic solutions of the Korteweg–de Vries equation subject to additive noise is investigated using numerical techniques. Various types of additive white Gaussian noise are considered. The averaged solution amplitudes exhibit in all cases algebraic decay, verifying Wadati's universality conjecture. If the noise is time dependent, or position and time dependent, algebraic decay is obtained for intermediate times too. These intermediate-time results agree well with the outcome of an experiment on ion-acoustic soliton propagation in a noisy plasma. The distribution of soliton first passage times in a noisy medium is also discussed. [S1063-651X(98)10509-3]

PACS number(s): 05.40.+j, 52.35.Mw

I. INTRODUCTION

The remarkable stability of the solitonic solutions exhibited by certain nonlinear differential equations was discovered by Zabusky and Kruskal in their classical numerical study [1]. They showed that Korteweg–de Vries (KdV) solitons preserve their identities through soliton-soliton interactions. The only signature left by these interactions is a phase shift. Due to the increasing applicability of solitons to practical problems, attention has recently been turned to the stability of solitons subject to random perturbations. The extensive reviews by Bass and co-workers [2] and by Abdullaev [3] describe many of the results obtained in this area. On the other hand, the possibility of observing Anderson localization effects on nonlinear excitations in disordered media has also led to interesting research on the interplay between nonlinearity and disorder [4].

The KdV equation is still a paradigm for soliton-bearing nonlinear differential equations. As such, it has been the subject of many papers devoted to the analysis of the influence of various types of random perturbations on its solutions. Abdullaev and co-workers have investigated the evolution of randomly perturbed initial solitonic states, assuming that this evolution is controlled by the nonrandom KdV equation [5]. Other authors have studied the evolution of an initially deterministic wave form due to a stochastically perturbed KdV equation. By using the inverse scattering technique and a suitable moving reference frame, Wadati was able to obtain the exact one-soliton solution for additive time-dependent white Gaussian noise [6]. He showed that the average of the single-soliton solution over many realizations should behave as a Gaussian whose width increases as $t^{3/2}$ at long times. Thus the soliton performs a superdiffusive motion in addition to its constant-speed displacement. Concomitantly, he predicted that the Gaussian amplitude should decrease as $t^{-3/2}$. Later, Wadati and Akutsu extended this work to obtain exact multisolitonic solutions for additive time-dependent noise, and to investigate the influence of dissipation [7]. They showed that dissipation leads to normal soliton diffusion: the ensemble average is a Gaussian whose width increases asymptotically only as $t^{1/2}$. On the other hand, there is no ''mass conservation'' and the overall amplitude is exponentially damped. Their results were rederived by Herman [8], who also considered multiplicative noise, in the cases corresponding to dissipation and velocity fluctuations. For the averaged solitons, Herman obtained Gaussians whose widths grew with the same time dependence Wadati had obtained for the purely time-dependent additive noise. Other consequences of perturbing the solitons with multiplicative noise, such as the generation of radiation, were analyzed in Ref. [2]. More recently, Iizuka discussed the diffusion of solitons under the effect of multiplicative noise with long-range correlations [9], concluding that the soliton diffusion must be anomalous if the correlations decay algebraically with an exponent smaller than unity.

Wadati also conjectured that the algebraic decay ($\sim t^{-\alpha}$) of the amplitude with time in the long-time regime should have a universal character. In this connection, the problem of constructing integrable stochastic systems was analyzed in a recent book by Konotop and Vázquez [10]. These authors found that, by changing to a proper reference frame, an additive noise can be transformed into a multiplicative noise plus a fluctuating background. In this way, it can be shown that Wadati's universality conjecture is verified, with an exponent α that depends on the statistics and on the form of the random term used. However, although the problem is integrable or nearly integrable, it is not always possible to find the exponent explicitly, except by the use of approximate perturbation methods [11]. In this paper we determine α numerically for nearly integrable KdV systems, e.g., with various types of low-intensity noise. Moreover, we extend the analysis to the intermediate-time region, which, to our knowledge, has never been studied.

Since experimentalists often probe intermediate times, it is useful to have explicit predictions that are not restricted to the asymptotic regime. Frequently, analytical results for

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these intermediate times are either unattainable or too complex to be easily applied. Numerical solutions or simulations then provide the sole possible standards against which experimental results can be compared. In particular, we will show that the hitherto unexplained results of the experiment of Chang and co-workers [12] on the propagation of ion-acoustic solitons in a nonquiescent plasma agree very well with the results obtained for position- and time-dependent noise in the intermediate time regime.

We have found no studies of the cases in which the KdV equation is modified by additive noise that depends on position or on both time and position. This problem is more realistic than that of the noise depending on time alone, but it is much harder to investigate using analytical techniques. In this paper we perform a detailed numerical analysis of the effects of various types of additive noise on the evolution of the individual soliton solutions and of their averages.

Although first-passage problems for particles moving in disordered systems have received a lot of attention, there are, as far as we know, no analyses of soliton first-passage times in a noisy medium. In this paper we propose an ansatz for the distribution of first passage times when the noise depends on time alone, and we use this ansatz in combination with numerical solutions to investigate the passage times for two interacting solitons.

We start Sec. II by revisiting Wadati's results for the t-dependent noise. We then present a simple derivation of the long-time result for the mean square displacement and examine the first-passage problem. Some interesting features of the unaveraged problem, which had previously received little attention, are also discussed. In Sec. III we report the results of the numerical simulations (some preliminary results, for shorter runs, were reported in Ref. [13]). Our conclusions are summarized in Sec. IV.

II. THE KdV EQUATION WITH ADDITIVE NOISE

The equation we consider in this paper is the KdV equation subject to additive noise and dissipation [7,8],

$$u_t - 6uu_x + u_{xxx} + \gamma u = \eta(x,t), \tag{1}$$

where $u(x,t)$ is a real field, $\eta(x,t)$ an external random force, and the subscripts x and t stand for the partial derivatives with respect to position and time, respectively. The damping coefficient γ is non-negative. We choose $\eta(x,t)$ to be a white Gaussian noise, whose statistical averages satisfy $\langle \eta(x,t)\rangle = 0$ and

$$\langle \eta(x,t)\,\eta(x',t')\rangle = 2\varepsilon\,\delta(x-x')\,\delta(t-t'), \tag{2}$$

where ε characterizes the noise intensity.

The well-known soliton solutions of the deterministic KdV equation ($\gamma=\eta=0$) have the form

$$u(x,t) = 2\kappa^2 \ \mathrm{sech}^2[\kappa(x-x_0) - 4\kappa^3 t], \tag{3}$$

where x_0 gives the initial soliton location.

Let us start by considering the influence of noise in the absence of dissipative effects, i.e., when $\gamma=0$. Even with this simplification, the problem cannot be solved analytically unless $\eta=\eta(t)$ alone. In this case, Wadati found that the unaveraged one-soliton solution has the form [6]

$$u(x,t) = W(t) + 2\kappa^2 \ \mathrm{sech}^2 \times\left[\kappa(x-x_0) - 4\kappa^3 t + 6\kappa \int_0^t W(t')\,dt'\right], \tag{4}$$

where

$$W(t) = \int_0^t \eta(t')\,dt'. \tag{5}$$

Wadati also computed the statistical average $\langle u(x,t)\rangle$, showing that, at short times, $t \ll t^* = (48\kappa^2\varepsilon)^{-1/3}$, it is still approximately given by the one-soliton solution (3). For $t \gg t^*$, the amplitude of $\langle u(x,t)\rangle$ decreases as $t^{-3/2}$, while its width δx increases as $t^{3/2}$. Wadati termed this phenomenon ''soliton diffusion.''

The long-time behavior can also be easily obtained by using a Langevin equation formalism. To see this, we note that in each experimental realization, although the soliton shape is unperturbed, its maximum x_M moves according to the equation

$$x_M(t) = x_0 + 4\kappa^2 t - 6\int_0^t W(t')\,dt'. \tag{6}$$

This means that a stochastic term is added to the uniform displacement of the soliton in such a way that

$$\ddot{x}_M(t) = -6\,\eta(t), \tag{7}$$

with $\langle \eta(t)\rangle = 0$ and $\langle \eta(t)\,\eta(t')\rangle = 2\varepsilon\,\delta(t-t')$.

To obtain an equation for the mean square displacement, it is convenient to add temporarily a small dissipative term. Omitting the subscript M, we write

$$\ddot{x}(t) + \Gamma \dot{x}(t) = -6\,\eta(t), \tag{8}$$

where Γ is the friction constant. This equation is easily integrated, leading to

$$x(t) - x(0) = \frac{1}{\Gamma}\left[4\kappa^2(1-e^{-\Gamma t}) - 6 \times \int_0^t d\tau\,\eta(\tau)(1-e^{-\Gamma(t-\tau)})\right], \tag{9}$$

where $4\kappa^2$ is the initial velocity. We can now compute the mean square displacement $(\delta x)^2 = \langle [x(t)-x(0)]^2\rangle$. At long times we obtain the usual diffusive behavior, $(\delta x)^2 \sim t$ (as found by Wadati and Akutsu [7]). However, by letting the friction coefficient $\Gamma \to 0$ first, we find

$$(\delta x)^2 = 16\kappa^4 t^2 + 24\varepsilon t^3. \tag{10}$$

The first term on the right-hand side corresponds to the uniform displacement with the unperturbed soliton speed, while the second term gives the noise-induced superdiffusive behavior, which eventually prevails.

We now examine the problem of the soliton first passage times (FPTs) in the case of the integrable model with a time-dependent noise. Since the soliton is an extended object, we must state precisely the meaning of the expression ''soliton passage time.'' We will say that a soliton passes through a given point when the maximum elongation of the disturbance reaches that point, i.e., the passage time t_P through point L is defined in each experiment through the equation $x_M(t_P)=L$. This criterion is not only intuitively appealing but it is also very easy to implement in the simulations.

Suppose that it is known that the soliton passes through the origin at time $t=0$. We would like to obtain information about how long it takes the soliton to reach the point $x=L$ for the first time. Although in the absence of noise the duration of this interval can be exactly calculated, once noise is introduced the soliton diffuses and all we can predict is the distribution of its FPTs. The probability density for the FPT in the case of normal diffusion of a point particle is discussed by Feller [14]. If the particle starts at the point $z>0$ at $t=0$, he obtains that the probability density for its FPT through the origin is given by

$$N(z,D,c;t)=\frac{z}{(4\pi Dt^3)^{1/2}}\,e^{-(z+ct)^2/4Dt}, \qquad (11)$$

where D is the diffusion coefficient and c is the drift speed. Equation (10) suggests that the probability density for the soliton FPT could be described by taking $z=L$, $ct=-4\kappa^2 t$, and $Dt=12\varepsilon t^3$ in Eq. (11), provided that we restrict its application to long times, $t\gg t^*$, and to $\eta=\eta(t)$. Therefore, we propose

$$N_1(L,\varepsilon,\kappa;t)=\frac{LR}{(48\pi\varepsilon t^5)^{1/2}}\,e^{-(L-4\kappa^2 t)^2/48\varepsilon t^3}, \qquad (12)$$

where R is a normalization constant. We have performed long-time simulations that confirm the validity of this ansatz for the superdiffusive soliton case. We will present some FPT distributions for the two-soliton problem in the next section.

Multisolitonic solutions subject to noise were studied by Wadati and Akutsu [7]. For any individual noise realization, the solitons preserve their shapes, while their locations are all identically affected by the noise. A series of snapshots of the solitons would show that the relative distances between their maxima grow linearly with time, as if there were no noise. This is important for signal propagation, since it means that the solitons maintain their relative positions as they move through the noisy medium. Our numerical simulations show that this is indeed the case for any given soliton pair. It must be remarked, however, that the knowledge that the leading soliton crossed a given lattice point at $t=0$ does not suffice to ascertain precisely the passage time of the second soliton: since the two-soliton system continues to perform a random walk at $t>0$, the exact first passage time of the trailing soliton cannot be predicted.

No theoretical solutions are available for the case $\eta=\eta(x)$. However, since the noise at a given location influences in exactly the same way all points on the passing wave, it is reasonable to expect a strong enhancement of its effects. Even small noise intensities will soon give rise to marked deformations in the soliton; it is therefore not surprising that Herman mentions that ''there are some divergence problems with this type of noise [8].'' The numerical solutions, on the other hand, are informative, whenever the noise is chosen to be weak enough. If $\eta=\eta(x,t)$ the effects of the noise are not so devastating; because of the time dependence, the influence of the noise at a given location x on different points on the wave tends to average out and the soliton solution can survive much higher noise intensities, as we will show in the next section.

III. RESULTS AND DISCUSSION

In this section we present the discussion of simulations performed to investigate the effects of various types of additive noise on the evolution of solitonic solutions. In the numerical work we used the centered finite-difference discretization scheme proposed by Zabusky and Kruskal [1]. The stability conditions for this scheme were analyzed in detail by Vliegenthart [15] and our particular implementation was tested by showing that the exact solitonic solutions preserved their form after very long runs. This simple scheme proved to be both fast and accurate; we performed runs on a parallel supercomputer (Connection Machine CM-5, Thinking Machine Corp.) and on a sequential machine (SUN, Ultra 1). In the simulations we have let $t\to t\tau$, where $\tau=0.000\,12$ is the time discretization step and t is the number of time steps. We have taken the length of the space discretization step to be $\lambda=0.1$ and the initial single-soliton amplitude to be $A(t=0)=2\kappa^2=2$.

In order to understand to what extent solitonic properties are preserved when noise is introduced, we start by presenting (Fig. 1) an overview of the influence of noise on three situations that characterize solitonic solutions: single-soliton propagation (left column), soliton separation of an exact two-soliton initial condition (central column), and soliton generation out of a nonsolitonic initial condition (right column). In the figure we show the effects of three types of additive stochastic perturbations, $\eta(t)$, $\eta(x,t)$, and $\eta(x)$ on *individual* runs; we have superimposed snapshots taken at two different times, the $t=0$ snapshot corresponding to the initial condition. As a reference, we report the results for the deterministic case ($\eta=0$) in the first row. We present the results for $\eta=\eta(t)$ in the second row, those for $\eta=\eta(x,t)$ in the third row, and those for $\eta=\eta(x)$ in the fourth row. The respective noise intensities are indicated in the vertical axes labels. All plots were made using a reference frame that moves with the speed of the linear waves. From the first column we see that the noise does not destroy the soliton, even though it modifies its propagation speed, as we can conclude from the shift in the peak position with respect to that corresponding to the unperturbed case. As expected, $\eta(t)$ generates a vertical shift of the soliton, while in the other cases stronger fluctuations are present and, for long times and strong noise, may even mask the soliton completely. The two-soliton initial condition still gives rise to two solitonlike waves, although in the $\eta(x)$ case trailing shelves appear behind each solitonlike solution. Note that, for the particular run represented here, the leading soliton is well ahead of its ''unperturbed'' position, a consequence of the high efficacy of the $\eta(x)$ noise. For $\eta(t)$ we also verified that the distance between solitons grows linearly in time, exactly as in the absence of noise. The last column shows that the generation of multiple solitonlike solutions out of a steplike pulse is not affected by the noise. Although this

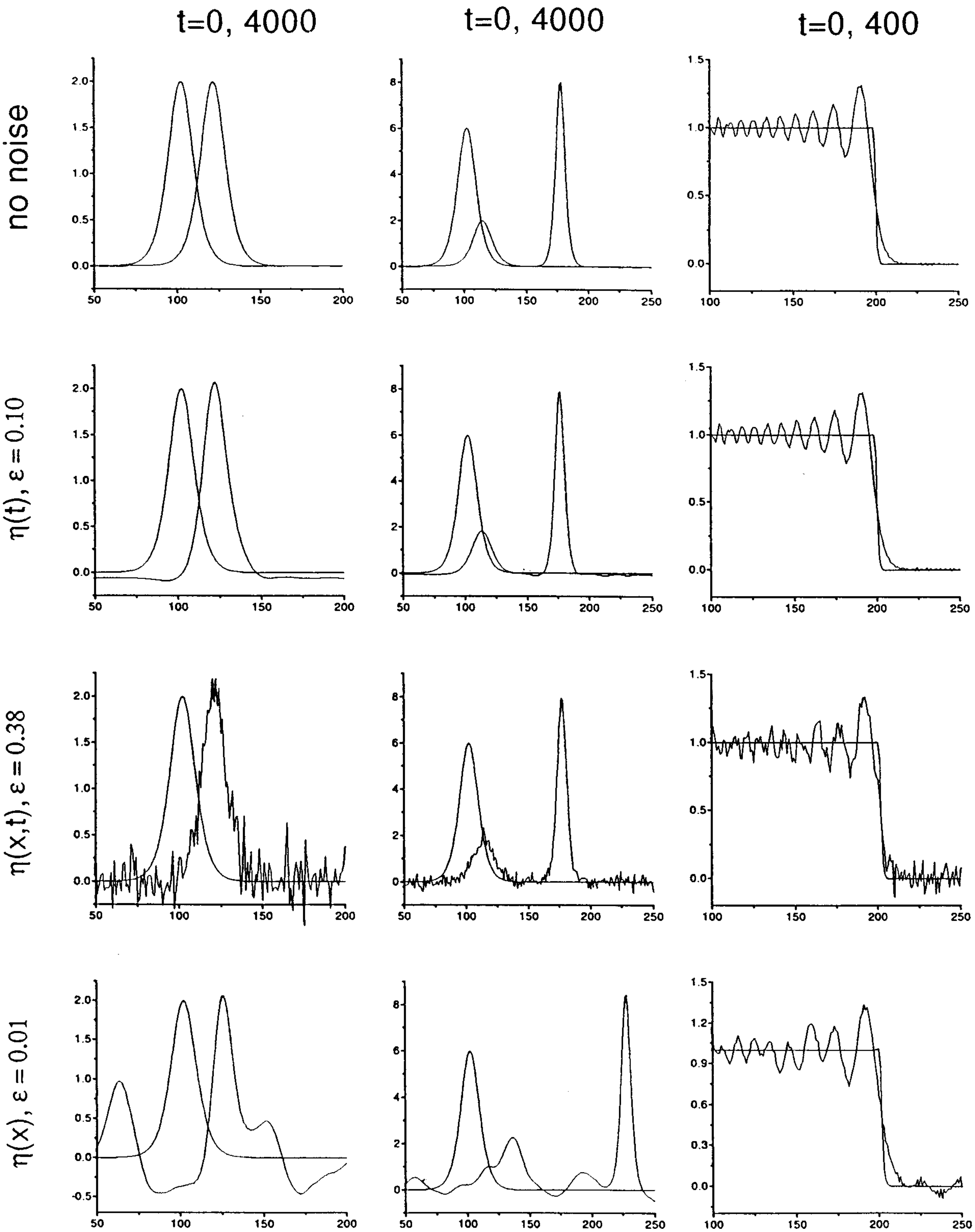

FIG. 1. Overview of the effects of various types of additive noise on soliton propagation and generation. Abscissa labels give the location of spatial nodes. Two snapshots of single runs taken at different times are shown in each case. The first (reference) row corresponds to the unperturbed problem and the following rows exhibit the effects of the kinds of noise specified in the vertical labels. The first column depicts single soliton propagation, the second column corresponds to the emergence of separate solitons out of a two-soliton initial condition, and the third column shows multiple solitons being generated out of a rectangular shelf.

process can be described analytically for $\eta(t)$ [16] the simulation confirms its robustness against the other perturbation types.

Figures 2–5 show the statistical effects of noise. In all cases the averages were taken over 200 runs. No noticeable differences emerge if we add more runs. Within the limits of numerical precision the evolution of each sample was followed exactly and the averages were taken at the end. At

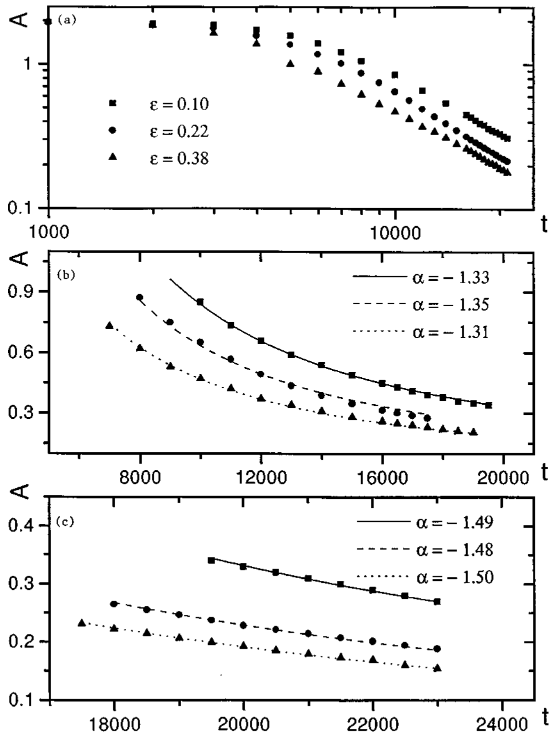

FIG. 2. Amplitudes of the Gaussian distribution of solitons for $\eta(t)$ and the noise intensities indicated in the figure. Abscissa units in Figs. 2–5 correspond to the number of time steps. In these figures, the ordinate units are chosen so that $A(t=0)=2$. (a) is a log-log plot extending up to 23 000 time steps. (b) and (c) are normal plots for the intermediate and long time regions, respectively. In these regions the average solution is well described by a Gaussian whose amplitude decays with two different power laws (which do not depend on noise intensity) indicated by the exponent α.

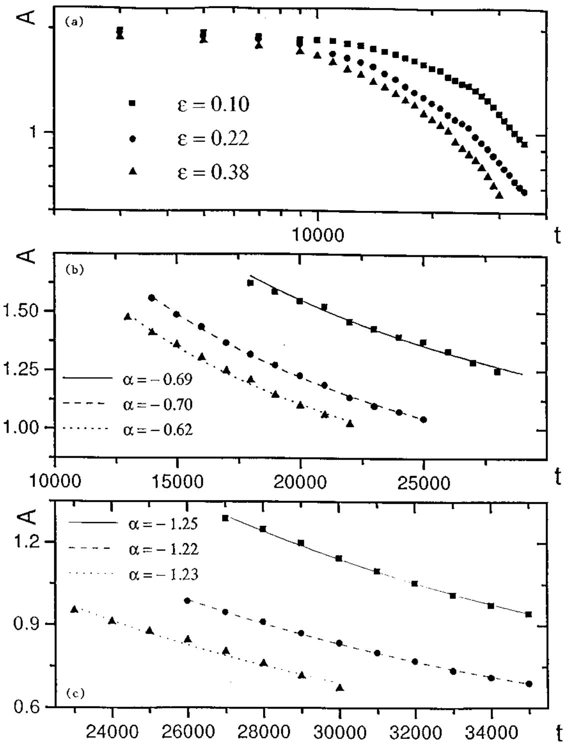

FIG. 3. Amplitudes of the Gaussian distribution of solitons for $\eta(x,t)$ and the noise intensities indicated in the figure. The simulation extends up to 36 000 time steps and the results are presented as in Fig. 2.

short times the averaged solution is not very different from the unperturbed single-soliton solution, Eq. (3). At longer times Gaussians provide very good fits for the averages. We have performed a detailed study of the widths and amplitudes of these Gaussians. Here we report only the results for the amplitudes, for which the fits are more precise. Let us begin by looking at the purely time-dependent noise $\eta=\eta(t)$, whose correlator satisfies $\langle \eta(t)\eta(t')\rangle = 2\varepsilon\,\delta(t-t')$. In Fig. 2 we consider three different noise intensities, as indicated. Figure 2(a) is a log-log plot that suggests the presence of three different regimes. At short times, a Gaussian does not provide a good fit, but for $t>8000$ (somewhat earlier in the case of $\varepsilon=0.38$) a Gaussian starts fitting well and its amplitude seems to decrease, following a power law. At the longest times studied, the amplitude of the Gaussian appears to decay, following a stronger power law. In Figs. 2(b) and 2(c) we have zoomed the results for the intermediate and long times, respectively, using linear scales. These plots indicate that in the intermediate region the amplitude decays approximately as $t^{-4/3}$, while Wadati's prediction of a $t^{-3/2}$ decay is clearly reproduced at the longest times. These exponents do not seem to depend on noise intensity, in agreement with Wadati's universality hypothesis. We also observe that the amplitude decreases faster and that the onset of the asymptotic regime occurs earlier for higher noise intensities. This is as it should be expected from Wadati's calculation; indeed, using his t^* [see below Eq. (5)] and the selected time scale, the characteristic time corresponds to $t^*\tau^{-1}=3170$ time steps for $\varepsilon=0.38$ and to 4940 time steps for $\varepsilon=0.1$. In all cases, we checked that the total ''soliton mass,'' i.e., the integral of $u(x,t)$ over all x, is conserved.

In Fig. 3 we show the results obtained by using position- and time-dependent noise $\eta(x,t)$. The noise intensities and the plotting strategy are the same as in Fig. 2. No analytical

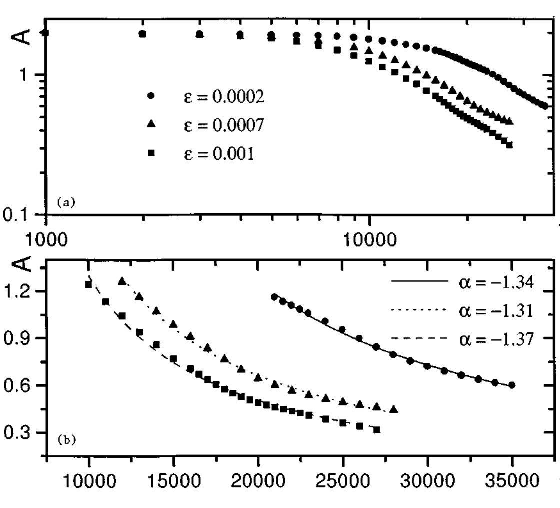

FIG. 4. Amplitudes of the Gaussian distribution of solitons for $\eta(x)$ and the noise intensities indicated in the figure. The intermediate algebraic decay region has disappeared, but asymptotically we get a power law decrease with an exponent close to $-4/3$.

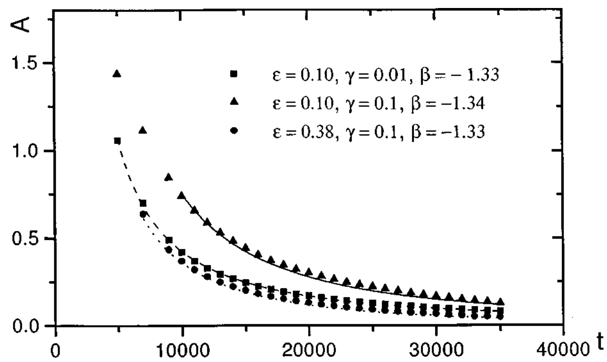

FIG. 5. Amplitudes of the Gaussian distribution of solitons with damping for $\eta(t)$ and the noise intensities indicated on the figure. The curves were fit with the function $t^{-\beta}\exp(-\frac{2}{3}\varepsilon\gamma t)$. The corresponding values of β and γ are also indicated on the figure.

predictions are available for this problem, but the figures clearly indicate a time evolution that parallels that corresponding to the $\eta(t)$ case: a slower power law decay (now close to $t^{-2/3}$) at intermediate times and a faster power law decay (close to $t^{-5/4}$) at long times. The intermediate-time result is particularly meaningful, since it is in excellent agreement with the results obtained by Chang and coworkers for the propagation of ion-acoustic solitons in plasmas [12]. This can be seen by a direct comparison of their Fig. 6(b) and our Fig. 3(b): not only is the power law decay the same, but also the absolute magnitudes are of the same order. This suggests that (i) the experiment probed the intermediate-time region and not the asymptotic region, (ii) spatial fluctuations were relevant, and (iii) damping was relatively weak.

Next we studied the case of purely position-dependent noise. For the reasons indicated above, this type of noise has an accumulative effect that tends to degrade the soliton. Therefore, we chose noise amplitudes much smaller than those used before. As we see from Fig. 4, no clear power law decay is observed at intermediate times, but for long times the amplitude decreases approximately as $t^{-4/3}$. For times longer than those reported, the average solution becomes meaningless, since the solitons are completely submerged in the noise.

We also investigated the stochastic damped KdV equation. In Fig. 5 we report the results obtained using a time-dependent noise $\eta=\eta(t)$. The theoretical predictions of Refs. [7] and [8] indicate that the amplitude should decay asymptotically as $t^{-\beta}\exp(-\frac{2}{3}\varepsilon\gamma t)$, with $\beta=1/2$. Our numerical analysis shows that the amplitude indeed decays exponentially but we get very good fits with $\beta\approx 4/3$. It is possible that the simulations do not reach into the truly asymptotic regime, which is not observable. In fact, Herman's asymptotic form for the solution is probably valid only when the soliton amplitude is already so small that the corresponding numerical results are unreliable.

In the last part of this section we discuss some normalized histograms with soliton passage times through a fixed spatial point for the case of the two-soliton solution, with $\eta=\eta(t)$ and $\gamma=0$. The initial condition is the same as in the second column of Fig. 1. The FPT distributions for both solitons are presented in Fig. 6. The histograms were built using 1080 runs. Since in some of the realizations the delay of the slower solitons was very long, we have omitted a portion of the distribution tail in Figs. 6(b) and 6(c).

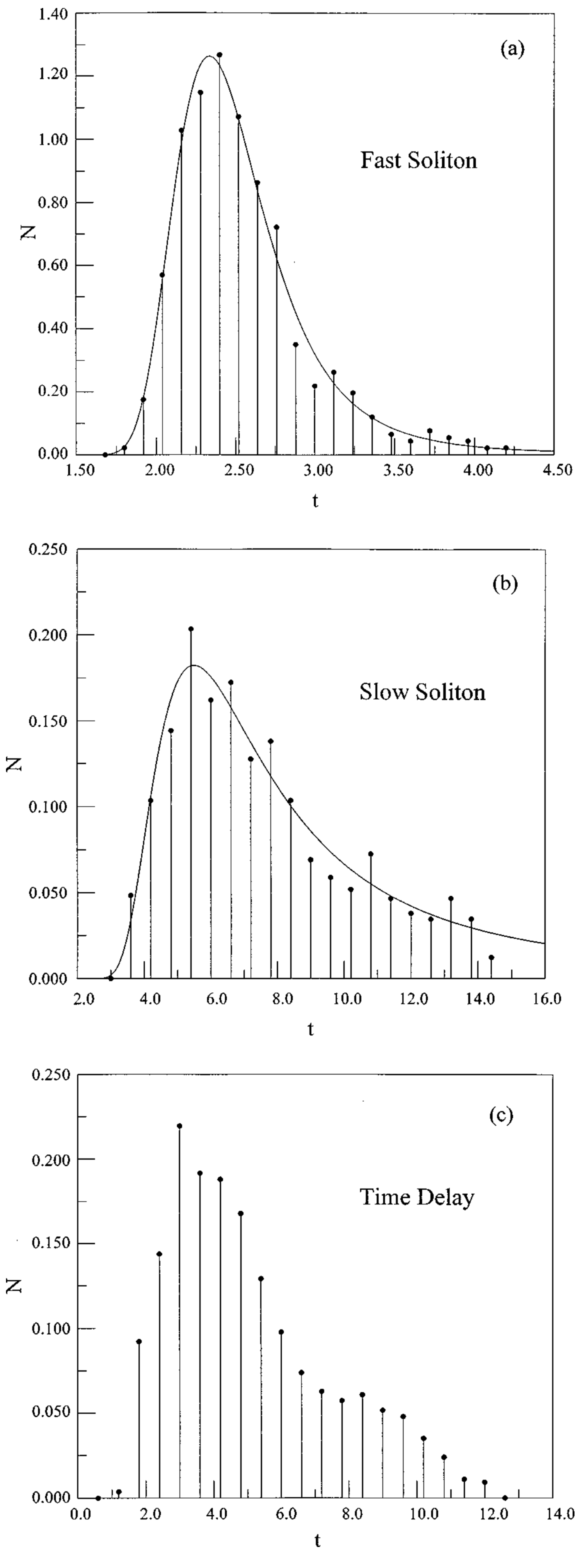

FIG. 6. Histograms for the passage times of two solitons through the node $i=675$ after starting from $i=300$ (the lattice has a total of 1200 nodes). The solitons start as an exact two-soliton solution, which begins to separate at $t=0$. Here $\eta=\eta(t)$ and $\varepsilon=0.1$.

Due to the soliton-soliton interaction, we cannot use Eq. (12) directly to describe the numerical results. However, we can slightly generalize this equation by adding the fitting parameters α (the width of the distribution of arrival times) and P (the phase shift generated by the interaction in a noisy medium). For each soliton we try

$$N_2(L,\alpha,P,\kappa;t)=\frac{LR}{(48\pi\alpha t^5)^{1/2}}\,e^{-(L-4\kappa^2 t-P)^2/48\alpha t^3}, \tag{13}$$

an equation that should hold after the solitons have been resolved. Of course, both α and P will depend on L (in the case of a single soliton, we would have $\alpha=\varepsilon$). For simplicity, and since the distribution tail was cut off, we obtained the normalization constant R directly from the fit. As it can be seen from Figs. 6(a) and 6(b), Eq. (13) describes very well the FPT distributions for both the leading and the trailing solitons, for which $\kappa=2$ and $\kappa=1$, respectively. Note that in this figure we have used the ''real'' time in the abscissa; the distance from the origin to the recording point is $L=(675-300)\lambda=37.5$. We found that $\alpha=0.0799$ and $P=-1.424$ were the best-fit parameters for the fast-soliton distribution, while $\alpha=0.1135$ and $P=-5.541$ were the best ones for the slow soliton. It appears that the interplay of noise and soliton-soliton interaction tends to concentrate the arrival times of the leading soliton and to separate the arrival times of the trailing one. This is not surprising, considering that it takes the trailing soliton a longer time to reach the target point and the integrated effects of the noise should be consequently more intense.

Trials performed with $\varepsilon=0$ reveal that $P>0$ for the faster soliton and $P<0$ for the slower soliton (which is in agreement with theoretical results [17]). The negative sign of the parameter P for *both* solitons when $\varepsilon>0$ indicates that the noise introduces an additional negative phase shift.

Although the distance between the solitons is completely deterministic for each run, we cannot predict the time delay (i.e., the separation between transit times). The reason is that both solitons continue their coupled random walks in the time intervening between their respective passages through the fixed point. A normalized histogram for the time delay is presented in Fig. 6(c).

IV. CONCLUSIONS

We have performed a detailed numerical study of the propagation of solitonic solutions of the Korteweg–de Vries equation in the presence of various types of noise. Our results establish the validity of Wadati's ''universality'' conjecture about the asymptotic behavior of the averages for the KdV problem; we also obtain precise values for the exponent characterizing the decay. These exponents depend on the nature of the noise but not on its intensity. We note that we recently confirmed Wadati's conjecture for the case of the solutions of Boussinesq's equation propagating under the influence of additive time-dependent noise [18]; the exponent obtained in Ref. [18] agrees with the corresponding exponent ($\alpha=1.5$) for the KdV equation. We have thus confirmed Wadati's conjecture on a restricted scale; more work should be done to check its validity for the solutions of other soliton-bearing equations. When dissipation is included, we confirm the prediction of an exponential decay modified by a power law factor, but we obtain a power that is stronger than that predicted by the available theory. We believe that the reason for this is that in the theoretically predicted asymptotic regime the attenuation has already caused a strong reduction of the average, which becomes completely unobservable due to the noise.

Numerical solutions have the advantage of leading to predictions for all times. We have shown that when $\eta=\eta(t)$ or $\eta=\eta(x,t)$, there is an intermediate time range for which algebraic decay is to be expected, with a smaller exponent than in the asymptotic regime. The agreement between our intermediate-time solution and the results of the experiment of Chang *et al.* suggests that these were not completely understood because the experiment had not probed the truly asymptotic region, the only one for which easy-to-interpret analytical predictions were then available. Finally, we examined some statistical properties of the soliton first passage times in a noisy medium, a subject that, as far as we know, had never been investigated before. An ansatz generalizing Feller's formula for the first passage time distribution was proposed and verified numerically. A more detailed analysis of this problem is in progress.

ACKNOWLEDGMENTS

This work was partially supported by the National Science Foundation through Grant No. HRD-9450342. We are also grateful to the Ministero dell'Universitá e della Ricerca Scientifica e Tecnologica (Italy) for the grant that allowed the cooperation between the Politecnico di Torino and the University of Puerto Rico. We wish to thank Professor P. P. Delsanto for many useful discussions.

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